

GRAPH ORIENTATIONS AND LINEAR EXTENSIONS.

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Abstract. Given an underlying undirected simple graph, we consider the set of all acyclic orientations of its edges. Each of these orientations induces a partial order on the vertices of our graph and, therefore, we can count the number of linear extensions of these posets. We want to know which choice of orientation maximizes the number of linear extensions of the corresponding poset, and this problem will be solved essentially for comparability graphs and odd cycles, presenting several proofs. The corresponding enumeration problem for arbitrary simple graphs will be studied, including the case of random graphs; this will culminate in 1) new bounds for the volume of the stable polytope and 2) strong concentration results for our main statistic and for the graph entropy, which hold true *a.s.* for random graphs. We will then argue that our problem springs up naturally in the theory of graphical arrangements and graphical zonotopes.

1. Introduction.

Linear extensions of partially ordered sets have been the object of much attention and their uses and applications remain increasing. Their number is a fundamental statistic of posets, and they relate to ever-recurring problems in computer science due to their role in sorting problems. Still, many fundamental questions about linear extensions are unsolved, including the well-known $1/3$ - $2/3$ *Conjecture*. Efficiently enumerating linear extensions of certain posets is difficult, and the general problem has been found to be $\sharp P$ -complete in Brightwell and Winkler (1991).

Directed acyclic graphs, and similarly, acyclic orientations of simple undirected graphs, are closely related to posets, and their problem-modeling values in several disciplines, including the biological sciences, needs no introduction. We propose the following problem:

Problem 1.1. Suppose that there are n individuals with a known contagious disease, and suppose that we know which pairs of these individuals were in the same location at the same time. Assume that at some initial points, some of the individuals fell ill, and then they started infecting other people and so forth, spreading the disease until all n of them were infected. Then, assuming no other knowledge of the situation, what is the most likely way in which the disease spread out?

Suppose that we have an underlying connected undirected simple graph $G = G(V, E)$ with n vertices. If we first pick uniformly at random a bijection $f : V \rightarrow [n]$, and then orient the edges of E so that for every $\{u, v\} \in E$ we select (u, v) (read u

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directed to v) whenever $f(u) < f(v)$, we obtain an acyclic orientation of E . In turn, each acyclic orientation induces a partial order on V in which $u < v$ if and only if there is a directed path $(u, u_1), (u_1, u_2), \dots, (u_k, v)$ in the orientation. In general, several choices of f above will result in the same acyclic orientation. However, the most likely acyclic orientations so obtained will be the ones whose induced posets have the maximal number of linear extensions, among all posets arising from acyclic orientations of E . Our problem then becomes that of deciding which acyclic orientations of E attain this optimality property of maximizing the number of linear extensions of induced posets. This problem, referred to throughout this article as the *main problem for G* , was raised by Saito (2007) for the case of trees, yet, a solution for the case of bipartite graphs had been obtained already by Stachowiak (1988). The main problem brings up the natural associated enumerative question: For a graph G , what is the maximal number of linear extensions of a poset induced by an acyclic orientation of G ? This statistic for simple graphs will be herein referred to as the *main statistic* (Definition 3.12).

The central goal of this initial article on the subject will be to begin a rigorous study of the main problem from the points of view of structural and enumerative combinatorics. We will introduce **1)** techniques to find optimal orientations of graphs that are provably correct for certain families of graphs, and **2)** techniques to estimate the main statistic for more general classes of graphs and to further understand aspects of its distribution across all graphs.

In Section 2, we will present an elementary approach to the main problem for both bipartite graphs and odd cycles. This will serve as motivation and preamble for the remaining sections. In particular, in Section 2.1, a new solution to the main problem for bipartite graphs will be obtained, different to that of Stachowiak (1988) in that we explicitly construct a function that maps injectively linear extensions of non-optimal acyclic orientations to linear extensions of an optimal orientation. As we will observe, optimal orientations of bipartite graphs are precisely the *bipartite orientations* (Definition 2.1). Then, in Section 2.2, we will extend our solution for bipartite graphs to odd cycles, proving that optimal orientations of odd cycles are precisely the *almost bipartite orientations* (Definition 2.7).

In Section 3, we will introduce two new techniques, one geometrical and the other poset-theoretical, that lead to different solutions for the case of comparability graphs. Optimal orientations of comparability graphs are precisely the *transitive orientations* (Definition 3.2), a result that generalizes the solution for bipartite graphs. The techniques developed on Section 3 will allow us to re-discover the solution for odd cycles and to state inequalities for the general enumeration problem in Section 4. The recurrences for the number of linear extensions of posets presented in Corollary 3.11 had been previously established in Edelman et al. (1989) using *promotion and evacuation* theory, but we will obtain them independently as by-products of certain network flows in Hasse diagrams. Notably, Stachowiak (1988) had used some instances of these recurrences to solve the main problem for bipartite graphs.

Further on, in Section 4, we will also consider the enumeration problem for the case of random graphs with distribution $G_{n,p}$, $0 < p < 1$, and obtain tight concentration results for our main statistic, across all graphs. Incidentally, this will lead to new inequalities for the volume of the stable polytope and to a very strong

concentration result for the graph entropy (as defined in Csiszár et al. (1990)), which hold *a.s.* for random graphs.

Lastly, in Section 5, we will show that the main problem for a graph arises naturally from the corresponding graphical arrangement by asking for the regions with maximal *fractional volume* (Proposition 5.2). More surprisingly, we will also observe that the solutions to the main problem for comparability graphs and odd cycles correspond to certain vertices of the corresponding graphical zonotopes (Theorem 5.3).

Convention 1.2. *Let $G = G(V, E)$ be a simple undirected graph. Formally, an orientation O of E (or G) is a map $O : E \rightarrow V^2$ such that for all $e := \{u, v\} \in E$, we have $O(e) \in \{(u, v), (v, u)\}$. Furthermore, O is said to be acyclic if the directed graph on vertex set V and directed-edge set $O(E)$ is acyclic. On numerous occasions, we will somewhat abusively also identify an acyclic orientation O of E with the set $O(E)$, or with the poset that it induces on V , doing this with the aim to reduce extensive wording.*

When defining posets herein, we will also try to make clear the distinction between the ground set of the poset and its order relations.

2. Introductory results.

2.1. The case of bipartite graphs.

The goal of this section is to present a combinatorial proof that the number of linear extensions of a bipartite graph G is maximized when we choose a *bipartite orientation* for G . Our method is to find an injective function from the set of linear extensions of any fixed acyclic orientation to the set of linear extensions of a bipartite orientation, and then to show that this function is not surjective whenever the initial orientation is not bipartite. Throughout the section, let G be bipartite with $n \geq 1$ vertices.

Definition 2.1. *Suppose that $G = G(V, E)$ has a bipartition $V = V_1 \sqcup V_2$. Then, the orientations that either choose (v_1, v_2) for all $\{v_1, v_2\} \in E$ with $v_1 \in V_1$ and $v_2 \in V_2$, or (v_2, v_1) for all $\{v_1, v_2\} \in E$ with $v_1 \in V_1$ and $v_2 \in V_2$, are called bipartite orientations of G .*

Definition 2.2. *For a graph G on vertex set V with $|V| = n$, we will denote by $\text{Bij}(V, [n])$ the set of bijections from V to $[n]$.*

As a training example, we consider the case when we transform linear extensions of one of the bipartite orientations into linear extensions of the other bipartite orientation. We expect to obtain a bijection for this case.

Proposition 2.3. *Let $G = G(V, E)$ be a simple connected undirected bipartite graph, with $n = |V|$. Let O_{down} and O_{up} be the two bipartite orientations of G . Then, there exists a bijection between the set of linear extensions of O_{down} and the set of linear extensions of O_{up} .*

Proof. Consider the automorphism rev of the set $\text{Bij}(V, [n])$ given by $\text{rev}(f)(v) = n+1-f(v)$ for all $v \in V$ and $f \in \text{Bij}(V, [n])$. It is clear that $(\text{rev} \circ \text{rev})(f) = f$. However, since $f(u) > f(v)$ implies $\text{rev}(f)(u) < \text{rev}(f)(v)$, then rev reverses all directed paths in any f -induced acyclic orientation of G , and in particular the restriction of rev to the set of linear extensions of O_{down} has image O_{up} , and viceversa.

□

We now proceed to study the case of general acyclic orientations of the edges of G . Even though similar in flavour to Proposition 2.3, our new function will not in general correspond to the function presented in the proposition when restricted to the case of bipartite orientations.

To begin, we define the main automorphisms of $\text{Bij}(V, [n])$ that will serve as building blocks for constructing the new function.

Definition 2.4. Consider a simple graph $G = G(V, E)$ with $|V| = n$. For different vertices $u, v \in V$, let rev_{uv} be the automorphism of $\text{Bij}(V, [n])$ given by the following rule: For all $f \in \text{Bij}(V, [n])$, let

$$\begin{aligned} \text{rev}_{uv}(f)(u) &= f(v), \\ \text{rev}_{uv}(f)(v) &= f(u), \\ \text{rev}_{uv}(f)(w) &= f(w) \text{ if } w \in V \setminus \{u, v\}. \end{aligned}$$

It is clear that $(\text{rev}_{uv} \circ \text{rev}_{uv})(f) = f$ for all $f \in \text{Bij}(V, [n])$. Moreover, we will need the following technical observation about rev_{uv} .

Observation 2.5. Let $G = G(V, E)$ be a simple graph with $|V| = n$ and consider a bijection $f \in \text{Bij}(V, [n])$. Then, if for some $u, v, x, y \in V$ with $f(u) < f(v)$ we have that $\text{rev}_{uv}(f)(x) > \text{rev}_{uv}(f)(y)$ but $f(x) < f(y)$, then $f(u) \leq f(x) < f(y) \leq f(v)$ and furthermore, at least one of $f(x)$ or $f(y)$ must be equal to one of $f(u)$ or $f(v)$.

Let us present the main result of this section, obtained based on the interplay between acyclic orientations and bijections in $\text{Bij}(V, [n])$.

Theorem 2.6. Let $G = G(V, E)$ be a connected bipartite simple graph with $|V| = n$, and with bipartite orientations O_{down} and O_{up} . Let also O be an acyclic orientation of G . Then, there exists an injective function Θ from the set of linear extensions of O to the set of linear extensions of O_{up} and furthermore, Θ is surjective if and only if $O = O_{\text{up}}$ or $O = O_{\text{down}}$.

Proof. Let f be a linear extension of O , and without loss of generality assume that $O \neq O_{\text{up}}$. We seek to find a function Θ that transforms f into a linear extension of O_{up} injectively. The idea will be to describe how Θ acts on f as a composition of automorphisms of the kind presented in Definition 2.4. Now, we will find the terms of the composition in an inductive way, where at each step we consider the underlying configuration obtained after the previous steps. In particular, the choice of terms in the composition will depend on f . The inductive steps will be indexed using a positive integer variable k , starting from $k = 1$, and at each step we will know an acyclic orientation O_k of G , a set B_k and a function f_k . The set $B_k \subseteq V$ will always be defined as the set of all vertices incident to an edge whose orientation in O_k and O_{up} differs, and f_k will be a particular linear extension of O_k that we will define.

Initially, we set $O_1 = O$ and $f_1 = f$, and calculate B_1 . Now, suppose that for some fixed $k \geq 1$ we know O_k, B_k and f_k , and we want to compute O_{k+1}, B_{k+1} and f_{k+1} . If $B_k = \emptyset$, then $O_k = O_{\text{up}}$ and f_k is a linear extension of O_{up} , so we stop our recursive process. If not, then B_k contains elements u_k and v_k such that $f_k(u_k)$ and $f_k(v_k)$ are respectively minimum and maximum elements of $f_k(B_k) \subseteq [n]$. Moreover, $u_k \neq v_k$. We will then let $f_{k+1} := \text{rev}_{u_k v_k}(f_k)$, O_{k+1} be the acyclic orientation of G induced by f_{k+1} , and calculate B_{k+1} from O_{k+1} .

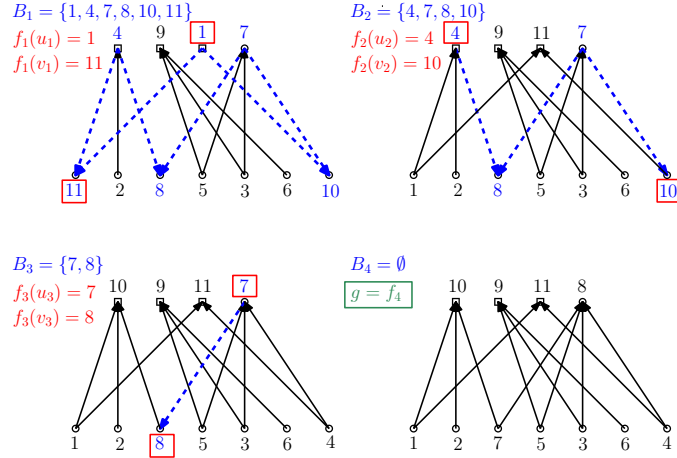


FIGURE 1. An example of the function Θ for the case of bipartite graphs. Squares show the numbers that will be flipped at each step, and dashed arrows indicate arrows whose orientation still needs to be reversed.

If we let m be the minimal positive integer for which $B_{m+1} = \emptyset$, then $\Theta(f) = (\text{rev}_{u_m v_m} \circ \dots \circ \text{rev}_{u_2 v_2} \circ \text{rev}_{u_1 v_1})(f)$. The existence of m follows from observing that $B_{k+1} \subsetneq B_k$ whenever $B_k \neq \emptyset$. In particular, if $B_k \neq \emptyset$, then $u_k, v_k \in B_k \setminus B_{k+1}$ and so $1 \leq m \leq \left\lfloor \frac{|B_1|}{2} \right\rfloor$. It follows that the pairs $\{u_k, v_k\}_{k \in [m]}$ are pairwise disjoint, $f(u_k) = f_k(u_k)$ and $f(v_k) = f_k(v_k)$ for all $k \in [m]$, and $f(u_1) < f(u_2) < \dots < f(u_m) < f(v_m) < \dots < f(v_2) < f(v_1)$. As a consequence, the automorphisms in the composition description of Θ commute. Lastly, f_{m+1} will be a linear extension of O_{up} and we stop the inductive process by defining $\Theta(f) = f_{m+1}$.

To prove that Θ is injective, note that given O and f_{m+1} as above, we can recover uniquely f by imitating our procedure to find $\Theta(f)$. Firstly, set $g_1 := f_{m+1}$ and $Q_1 := O_{\text{up}}$, and compute $C_1 \subseteq V$ as the set of vertices incident to an edge whose orientation differs in Q_1 and O . Assuming prior knowledge of Q_k, C_k and g_k , and whenever $C_k \neq \emptyset$ for some positive integer k , find the elements of C_k whose images under g_k are maximal and minimal in $g_k(C_k)$. By the discussion above and Observation 2.5, we check that these are respectively and precisely u_k and v_k . Resembling the previous case, we will then let $g_{k+1} := \text{rev}_{u_k v_k}(g_k)$, Q_{k+1} be the acyclic orientation of G induced by g_k , and compute C_{k+1} accordingly as the set of vertices incident to an edge with different orientation in Q_{k+1} and O . Clearly $g_{m+1} = f$, and the procedure shows that Θ is invertible in its image.

To establish that Θ is not surjective whenever $O \neq O_{\text{down}}$, note that then O contains a directed 2-path (w, u) and (u, v) . Without loss of generality, we may assume that the orientation of these edges in O_{up} is given by (w, u) and (v, u) . But then, a linear extension g of O_{up} in which $g(u) = n$ and $g(v) = 1$ is not in $\text{Im}(\Theta)$ since otherwise, using the notation and framework discussed above, there would exist different $i, j \in [m]$ such that $u_i = u$ and $v_j = v$, which then contradicts the choice of u_1 and v_1 . This completes the proof. \square

2.2. Odd cycles.

In this section $G = G(V, E)$ will be a cycle on $2n + 1$ vertices with $n \geq 1$. The case of odd cycles follows as an immediate extension of the case of bipartite graphs, but it will also be covered under a different guise in Section 4. As expected, the acyclic orientations of the edges of odd cycles that maximize the number of linear extensions resemble as much as possible bipartite orientations. This is now made precise.

Definition 2.7. *For an odd cycle $G = G(V, E)$, we say that an acyclic orientation of its edges is almost bipartite if under the orientation there exists exactly one directed 2-path, i.e. only one instance of (u, v) and (v, w) with $u, v, w \in V$.*

Theorem 2.8. *Let $G = G(V, E)$ be an odd cycle on $2n + 1$ vertices with $n \geq 1$. Then, the acyclic orientations of E that maximize the number of linear extensions are the almost bipartite orientations.*

First proof. Since the case when $n = 1$ is straightforward let us assume that $n \geq 2$, and consider an arbitrary acyclic orientation O of G . Again, our method will be to construct an injective function Θ' that transforms every linear extension of O into a linear extension of some fixed almost bipartite orientation of G , where the specific choice of almost bipartite orientation will not matter by the symmetry of G .

To begin, note that there must exist a directed 2-path in O , say (u, v) and (v, w) for some $u, v, w \in V$. Our goal will be to construct Θ' so that it maps into the set of linear extensions of the almost bipartite orientation O_{uvw} in which our directed path $(u, v), (v, w)$ is the unique directed 2-path. To find Θ' , first consider the bipartite graph G' with vertex set $V \setminus \{v\}$ and edge set $E \setminus (\{u, v\} \cup \{v, w\}) \cup \{u, w\}$, along with the orientation O' of its edges that agrees on common edges with O and contains (u, w) . Clearly O' is acyclic. If f is a linear extension of O , we regard the restriction f' of f to $V \setminus \{v\}$ as a strict order-preserving map on O' , and analogously to the proof of Theorem 2.6, we can transform injectively f' into a strict order-preserving map g' with $\text{Im}(g') = \text{Im}(f') = \text{Im}(f) \setminus \{f(v)\}$ of the bipartite orientation of G' that contains (u, w) . Now, if we define $g \in \text{Bij}(V, [n])$ via $g(x) = g'(x)$ for all $x \in V \setminus \{v\}$ and $g(v) = f(v)$, we see that g is a linear extension of O_{uvw} . We let $\Theta'(f) = g$.

The technical work for proving the general injectiveness of Θ' , and its non-surjectiveness when O is not almost bipartite, has already been presented in the proof of Theorem 2.6: That Θ' is injective follows from the injectiveness of the map transforming f' into g' , and then by noticing that $f(v) = g(v)$. Non-surjectiveness follows from noting that if O is not almost bipartite, then O contains a directed 2-path $(a, b), (b, c)$ with $a, b, c \in V$ and $b \neq v$, so we cannot have simultaneously $g'(a) = \min \text{Im}(f')$ and $g'(c) = \max \text{Im}(f')$. □

3. Comparability graphs.

In this section, we will study our main problem using more general techniques. As a consequence, we will be able to understand the case of comparability graphs, which includes bipartite graphs as a special case. Let us first recall the main object of this section:

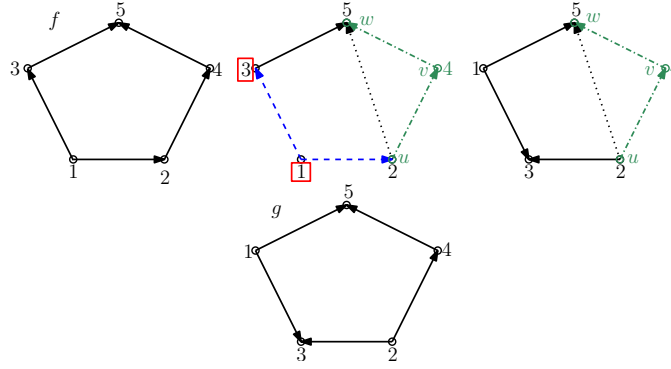


FIGURE 2. An example of the function Θ' for the case of odd cycles. Squares show the numbers that will be flipped at each step. Dashed arrows indicate arrows whose orientation still needs to be reversed, while dashed-dotted arrows indicate those whose orientation will never be reversed. In particular, 4 will remain labeling the same vertex during all steps.

Definition 3.1. A comparability graph is a simple undirected graph $G = G(V, E)$ for which there exists a partial order on V under which two different vertices $u, v \in V$ are comparable if and only if $\{u, v\} \in E$.

The acyclic orientations of the edges of a comparability graph G that maximize the number of linear extensions are precisely the orientations that induce posets whose comparability graph agrees with G .

Comparability graphs have been largely discussed in the literature, mainly due to their connection with partial orders and because they are perfectly orderable graphs and more generally, perfect graphs. Comparability graphs, perfectly orderable graphs and perfect graphs are all large hereditary classes of graphs. In Gallai's fundamental work in Gallai et al. (2001), a characterization of comparability graphs in terms of forbidden subgraphs was given and the concept of *modular decomposition* of a graph was introduced.

Note that, given a comparability graph $G = G(V, E)$, we can find at least two partial orders on V induced by acyclic orientations of E whose comparability graphs (obtained as discussed above) agree precisely with G , and the number of such posets depends on the modular structure of G . Let us record this idea in a definition.

Definition 3.2. Let $G = G(V, E)$ be a comparability graph, and let O be an acyclic orientation of E such that the comparability graph of the partial order of V induced by O agrees precisely with G . Then, we will say that O is a transitive orientation of G .

We will present two methods for proving our main result. The first one (Subsection 3.1) relies on Stanley's transfer map between the *order polytope* and the *chain polytope* of a poset, and the second one (Subsection 3.2) is made possible by relating our problem to *network flows*.

3.1. Geometry.

To begin, let us recall the main definitions and notation related to the first method.

Definition 3.3. We will consider \mathbb{R}^n with euclidean topology, and let $\{e_j\}_{j \in [n]}$ be the standard basis of \mathbb{R}^n . For $J \subseteq [n]$, we will define $e_J := \sum_{j \in J} e_j$ and $e_\emptyset := 0$; furthermore, for $x \in \mathbb{R}^n$ we will let $x_J := \sum_{j \in J} x_j$ and $x_\emptyset := 0$.

Definition 3.4. Given a partial order P on $[n]$, the order polytope of P is defined as:

$$\mathcal{O}(P) := \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \text{ and } x_j \leq x_k \text{ whenever } j \leq_P k, \forall i, j, k \in [n]\}.$$

The chain polytope of P is defined as:

$$\mathcal{C}(P) := \{x \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n] \text{ and } x_C \leq 1 \text{ whenever } C \text{ is a chain in } P\}.$$

Stanley's transfer map $\phi : \mathcal{O}(P) \rightarrow \mathcal{C}(P)$ is the function given by:

$$\phi(x)_i = \begin{cases} x_i - \max_{j <_P i} x_j & \text{if } i \text{ is not minimal in } P, \\ x_i & \text{if } i \text{ is minimal in } P. \end{cases}$$

Let P be a partial order on $[n]$. It is easy to see from the definitions that the vertices of $\mathcal{O}(P)$ are given by all the e_I with I an order filter of P , and those of $\mathcal{C}(P)$ are given by all the e_A with A an antichain of P .

Now, a well-known result of Stanley (1986) states that $\text{Vol}(\mathcal{O}(P)) = \frac{1}{n!}e(P)$ where $e(P)$ is the number of linear extensions of P . This result can be proved by considering the unimodular triangulation of $\mathcal{O}(P)$ whose maximal (closed) simplices have the form $\Delta_\sigma := \{x \in \mathbb{R}^n : 0 \leq x_{\sigma^{-1}(1)} \leq x_{\sigma^{-1}(2)} \leq \dots \leq x_{\sigma^{-1}(n)} \leq 1\}$ with $\sigma : P \rightarrow \mathbf{n}$ a linear extension of P . However, the volume of $\mathcal{C}(P)$ is not so direct to compute. To find $\text{Vol}(\mathcal{C}(P))$ Stanley made use of the transfer map ϕ , a pivotal idea that we now wish to describe in detail since it will provide a geometrical point of view on our main problem.

It is easy to see that ϕ is invertible and its inverse can be described by:

$$\phi^{-1}(x)_i = \max_{\substack{C \text{ chain in } P: \\ i \text{ is maximal in } C}} x_C, \text{ for all } i \in [n] \text{ and } x \in \mathcal{C}(P).$$

As a consequence, we see that $\phi^{-1}(e_A) = e_{A^\vee}$ for all antichains A of P , where A^\vee is the order filter of P induced by A . It is also straightforward to notice that ϕ is linear on each of the Δ_σ with σ a linear extension of P , by staring at the definition of Δ_σ . Hence, for fixed σ and for each $i \in [n]$, we can consider the order filters $A_i^\vee := \sigma^{-1}([i, n])$ along with their respective minimal elements A_i in P , and notice that $\phi(e_{A_i^\vee}) = e_{A_i}$ and also that $\phi(0) = 0$. From there, ϕ is now easily seen to be a unimodular linear map on Δ_σ , and so $\text{Vol}(\phi(\Delta_\sigma)) = \text{Vol}(\Delta_\sigma) = \frac{1}{n!}$. Since ϕ is invertible, without unreasonable effort we have obtained the following central result:

Theorem 3.5 (Stanley (1986)). *Let P be a partial order on $[n]$. Then, $\text{Vol}(\mathcal{O}(P)) = \text{Vol}(\mathcal{C}(P)) = \frac{1}{n!}e(P)$, where $e(P)$ is the number of linear extensions of P .*

Definition 3.6. Given a simple undirected graph $G = G([n], E)$, the stable polytope $\text{STAB}(G)$ of G is the full dimensional polytope in \mathbb{R}^n obtained as the convex hull of all the vectors e_I , where I is a stable (a.k.a. independent) set of G .

Now, the chain polytope of a partial order P on $[n]$ is clearly the same as the stable polytope $\text{STAB}(G)$ of its comparability graph $G = G([n], E)$ since antichains of P correspond to stable sets of G . In combination with Theorem 3.5, this shows that the number of linear extensions is a comparability invariant, i.e. two posets with isomorphic comparability graphs have the same number of linear extensions.

We are now ready to present the first proof of the main result for comparability graphs. We will assume connectedness of G for convenience in the presentation of the second proof.

Theorem 3.7. *Let $G = G(V, E)$ be a connected comparability graph. Then, the acyclic orientations of E that maximize the number of linear extensions are exactly the transitive orientations of G .*

First proof. Without loss of generality, assume that $V = [n]$. Let O be an acyclic orientation of G inducing a partial order P on $[n]$. If two vertices $i, j \in [n]$ are incomparable in P , then $\{i, j\} \notin E$. This implies that all antichains of P are stable sets of G , and so $\mathcal{C}(P) \subseteq \text{STAB}(G)$.

On the other hand, if O is not transitive, then there exists two vertices $k, \ell \in [n]$ such that $\{k, \ell\} \notin E$, but such that k and ℓ are comparable in P , i.e. the transitive closure of O induces comparability of k and ℓ . Then, $e_k + e_\ell$ is a vertex of the stable polytope $\text{STAB}(G)$ of G , but since $\mathcal{C}(P)$ is a subpolytope of the n -dimensional cube, $e_k + e_\ell \notin \mathcal{C}(P)$. We obtain that $\mathcal{C}(P) \neq \text{STAB}(G)$ if O is not transitive, and so $\mathcal{C}(P) \subsetneq \text{STAB}(G)$.

If O is transitive, then $\mathcal{C}(P) = \text{STAB}(G)$. This completes the proof. \square

3.2. Poset theory.

Let us now introduce the background necessary to present our second method. This will eventually lead to a different proof of Theorem 3.7.

Definition 3.8. *If we consider a simple connected undirected graph $G = G(V, E)$ and endow it with an acyclic orientation of its edges, we will say that our graph is an oriented graph and consider it a directed graph, so that every member of E is regarded as an ordered pair. We will use the notation $G_o = G_o(V, E)$ to denote an oriented graph defined in such a way, coming from a simple graph G .*

Definition 3.9. *Let $G_o = G_o(V, E)$ be an oriented graph. We will denote by \hat{G}_o the oriented graph with vertex set $\hat{V} := V \cup \{\hat{0}, \hat{1}\}$ and set of directed edges \hat{E} equal to the union of E and all edges of the form:*

$$\begin{aligned} & (v, \hat{1}) \text{ with } v \in V \text{ and } \text{outdeg}(v) = 0 \text{ in } G_o, \text{ and} \\ & (\hat{0}, v) \text{ with } v \in V \text{ and } \text{indeg}(v) = 0 \text{ in } G_o. \end{aligned}$$

A natural flow on G_o will be a function $f : \hat{E} \rightarrow \mathbb{R}_{\geq 0}$ such that for all $v \in V$, we have:

$$\sum_{(x,v) \in \hat{E}} f(x, v) = \sum_{(v,y) \in \hat{E}} f(v, y).$$

In other words, a natural flow on G_o is a nonnegative network flow on \hat{G}_o with unique source $\hat{0}$, unique sink $\hat{1}$, and infinite edge capacities.

First, let us relate natural flows on oriented graphs with linear extensions of induced posets.

Lemma 3.10. *Let $G_o = G_o(V, E)$ be an oriented graph with induced partial order P on V , and with $|V| = n$. Then, the function $g : \hat{E} \rightarrow \mathbb{R}_{\geq 0}$ defined by*

$$\begin{aligned} g(u, v) &= |\{\sigma : \sigma \text{ is a linear extension of } P \text{ and } \sigma(u) = \sigma(v) - 1\}| \\ &\quad \text{if } (u, v) \in E, \\ g(v, \hat{1}) &= |\{\sigma : \sigma \text{ is a linear extension of } P \text{ and } \sigma(v) = n\}| \\ &\quad \text{if } v \in V \text{ and } \text{outdeg}(v) = 0 \text{ in } G_o, \text{ and} \\ g(\hat{0}, v) &= |\{\sigma : \sigma \text{ is a linear extension of } P \text{ and } \sigma(v) = 1\}| \\ &\quad \text{if } v \in V \text{ and } \text{indeg}(v) = 0 \text{ in } G_o, \end{aligned}$$

is a natural flow on G_o . Moreover, the net g -flow from $\hat{0}$ to $\hat{1}$ is equal to $e(P)$.

Proof. Assume without loss of generality that $V = [n]$, and consider the directed graph K on vertex set $V(K) = [n] \cup \{\hat{0}, \hat{1}\}$ whose set $E(K)$ of directed edges consists of all:

$$\begin{aligned} (i, j) &\quad \text{for } i <_P j, \\ (i, j) \text{ and } (j, i) &\quad \text{for } i ||_P j, \\ (\hat{0}, i) &\quad \text{for } i \text{ minimal in } P, \text{ and} \\ (i, \hat{1}) &\quad \text{for } i \text{ maximal in } P. \end{aligned}$$

As directed graphs, we check that \hat{G}_o is a subgraph of K . We will define a network flow on K with unique source $\hat{0}$ and unique sink $\hat{1}$, expressing it as a sum of simpler network flows.

First, extend each linear extension σ of P to $V(K)$ by further defining $\sigma(\hat{0}) = 0$ and $\sigma(\hat{1}) = n + 1$. Then, let $f_\sigma : E(K) \rightarrow \mathbb{R}_{\geq 0}$ be given by

$$f_\sigma(x, y) = \begin{cases} 1 & \text{if } \sigma(x) = \sigma(y) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f_σ defines a network flow on K with source $\hat{0}$, sink $\hat{1}$, and total net flow 1, and then $f := \sum_{\sigma \text{ linear ext. of } P} f_\sigma$ defines a network flow on K with total net flow

$e(P)$. Moreover, for each $(x, y) \in \hat{E}$ we have that $f(x, y) = g(x, y)$. It remains now to check that the restriction of f to \hat{E} is still a network flow on \hat{G}_o with total flow $e(P)$.

We have to verify two conditions. First, for $i, j \in [n]$ and if $i ||_P j$, then

$$\begin{aligned} &|\{\sigma : \sigma \text{ is a lin. ext. of } P \text{ and } \sigma(i) = \sigma(j) - 1\}| \\ &= |\{\sigma : \sigma \text{ is a lin. ext. of } P \text{ and } \sigma(j) = \sigma(i) - 1\}|, \end{aligned}$$

so $f(i, j) = f(j, i)$, i.e. the net f -flow between i and j is 0. Second, again for $i, j \in [n]$, if $i <_P j$ but $i \not\leq_P j$, then $f(i, j) = 0$. These two observations imply that g defines a network flow on \hat{G}_o with total flow $e(P)$. \square

The next result was obtained in Edelman et al. (1989) using the theory of promotion and evacuation for posets, and their proof bears no resemblance to ours.

Corollary 3.11. *Let P be a partial order on V , with $|V| = n$. If A is an antichain of P , then $e(P) \geq \sum_{v \in A} e(P \setminus v)$, where $P \setminus v$ denotes the induced poset on $V \setminus \{v\}$. Similarly, if S is a cutset of P , then $e(P) \leq \sum_{v \in S} e(P \setminus v)$. Moreover, if I is a subset of V that is either a cutset or an antichain of P , then $e(P) = \sum_{v \in I} e(P \setminus v)$ if and only if I is both a cutset and an antichain of P .*

Proof. Let $G = G(V, E)$ be any graph that contains as a subgraph the Hasse diagram of P , and orient the edges of G so that it induces exactly P to obtain an oriented graph G_o . Let g be as in Lemma 3.10. Since edges representing cover relations of P are in G and are oriented accordingly in G_o , the net g -flow is $e(P)$. Moreover, by the *standard chain decomposition of network flows* of Ford Jr and Fulkerson (2010) (essentially Stanley's transfer map), which expresses g as a sum of positive flows through each maximal directed path of G_o , it is clear that for A an antichain of P , we have that $e(P) \geq \sum_{v \in A} \sum_{(x,v) \in \hat{E}} g(x, v)$, since antichains intersect maximal directed paths of G_o at most once. Similarly, for S a cutset of P , we have that $e(P) \leq \sum_{v \in S} \sum_{(x,v) \in \hat{E}} g(x, v)$ since every maximal directed path of G_o intersects S . Furthermore, equality will only hold in either case if the other case holds as well. But then, for each $v \in V$, the map **Trans** that transforms linear extensions of $P \setminus v$ into linear extensions of P and defined via: For σ a linear extension of $P \setminus v$ and $\kappa := \max_{y <_P v} \sigma(y)$,

$$\mathbf{Trans}(\sigma)(x) = \begin{cases} \kappa + 1 & \text{if } x = v, \\ \sigma(x) + 1 & \text{if } \sigma(x) > \kappa, \\ \sigma(x) & \text{otherwise,} \end{cases}$$

is a bijection onto its image, and the number $\sum_{(x,v) \in \hat{E}} g(x, v)$ is precisely $|\text{Im}(\mathbf{Trans})|$. \square

Getting ready for the second proof of Theorem 3.7, it will be useful to have a notation for the main object of study in this paper:

Definition 3.12. Let $G = G(V, E)$ be an undirected simple graph. The maximal number of linear extensions of a partial order on V induced by an acyclic orientation of E will be denoted by $\varepsilon(G)$.

Second proof of Theorem 3.7. Assume without loss of generality that $V = [n]$. We will do induction on n . The case $n = 1$ is immediate, so assume the result holds for $n - 1$. Note that every induced subgraph of G is also a comparability graph and moreover, every transitive orientation of G induces a transitive orientation on the edges of every induced graph of G . Now, let O be a non-transitive orientation of E with induced poset P , so that there exists a comparable pair $\{k, \ell\}$ in P that is stable in G . Let S be an antichain cutset of P . Then, S is a stable set of G . Letting $G \setminus i$ be the induced subgraph of G on vertex set $[n] \setminus \{i\}$, we obtain that $\varepsilon(G) \geq \sum_{i \in S} \varepsilon(G \setminus i) \geq \sum_{i \in S} e(P \setminus i) = e(P)$, where the first inequality is an application of Corollary 3.11 on a transitive orientation of G , along with Definition 3.12 and the inductive hypothesis, the second inequality is obtained after recognizing that the poset induced by O on each $G \setminus i$ is a subposet of $P \setminus i$ and by Definition 3.12, and the last equality follows because S is a cutset of P . If $|S| > 1$ or $S \cap \{k, \ell\} = \emptyset$, then by induction the second inequality will be strict. On the other hand, if $S = \{k\}$ or $S = \{\ell\}$, then the first inequality will be strict since $\{k, \ell\}$ is stable in G .

Lastly, the different posets arising from transitive orientations of G have in common that their antichains are exactly the stable sets of G , and their cutsets are exactly the sets that meet every maximal clique of G at least once, so by the corollary, the inductive hypothesis and our choice of S above, these posets have the same number of linear extensions and this number is in general at least $\sum_{i \in S} \varepsilon(G \setminus i)$, and strictly greater if $S = \{k\}$ or $S = \{\ell\}$.

□

4. Beyond comparability and enumerative results.

In this section, we will illustrate a short application of the ideas developed in Section 3 to the case of odd cycles, re-establishing Theorem 2.8 using a more elegant technique (Subsection 4.1) that applies to other families of graphs. Then, in Subsection 4.2, we will obtain basic enumerative results for $\varepsilon(G)$. Finally, in Subsection 4.3, we will study the random variable $\varepsilon(G)$ when G is a random graph with distribution $G_{n,p}$, $0 < p < 1$. As it will be seen, if $G \sim G_{n,p}$, then $\log_2 \varepsilon(G)$ concentrates tightly around its mean, and this mean is asymptotically equal to $n \log_2 \log_b n^2$, where $b = \frac{1}{1-p}$. This will permit us to obtain, for the case of random graphs, new bounds for the volumes of stable polytopes, and a very strong concentration result for the *entropy* of a graph, both of which will hold *a.s.*

4.1. A useful technique.

We start with two simple observations that remained from the theory of Section 3.

Firstly, note that for a general graph G , finding $\varepsilon(G)$ is equivalent to finding the chain polytope of maximal volume contained in $\text{STAB}(G)$, hence:

Observation 4.1. *For a simple graph G , we have:*

$$\varepsilon(G) \leq n! \text{Vol}(\text{STAB}(G)).$$

Also, directly from Theorem 3.7 we can say the following:

Observation 4.2. *Let P and Q be partial orders on the same ground set, and suppose that the comparability graph of P contains as a subgraph the comparability graph of Q . Then, $e(Q) \geq e(P)$ and moreover, if the containment of graphs is proper, then $e(Q) > e(P)$.*

Second proof of Theorem 2.8. Note that every acyclic orientation O of E induces a partial order on V whose comparability graph contains (as a subgraph) the comparability graph of a poset given by an almost bipartite orientation, and this containment is proper if O is not almost bipartite. By the symmetry of G , then all of the almost bipartite orientations are equivalent.

Note to proof: The same technique allows us to obtain results for other restrictive families of graphs, like odd cycles with isomorphic trees similarly attached to every element of the cycle or, perhaps more importantly, odd-anti-cycles, but we do not pursue this here.

□

4.2. General bounds for the main statistic.

Let us now turn our attention to the general enumeration problem. Firstly, we need to dwell on the case of comparability graphs, from where we will jump easily to general graphs.

Theorem 4.3. *Let $G = G(V, E)$ be a comparability graph, and further let $V = \{v_1, v_2, \dots, v_n\}$. For $u_1, u_2, \dots, u_k \in V$, let $G \setminus u_1 u_2 \dots u_k$ be the induced subgraph*

of G on vertex set $V \setminus \{u_1, u_2, \dots, u_k\}$. Then,

$$\varepsilon(G) \geq \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{\chi(G)\chi(G \setminus v_{\sigma 1})\chi(G \setminus v_{\sigma 1}v_{\sigma 2})\chi(G \setminus v_{\sigma 1}v_{\sigma 2}v_{\sigma 3}) \dots \chi(v_{\sigma n})},$$

where \mathfrak{S}_n denotes the symmetric group on $[n]$ and χ denotes the chromatic number of the graph.

Proof. Let us first fix a perfect order ω of the vertices of G , i.g. ω can be a linear extension of a partial order on V whose comparability graph is G . Let H be an induced subgraph of G with vertex set $V(H)$ and edge set $E(H)$, let ω_H be the restriction of ω to $V(H)$, and let Q be the partial order of $V(H)$ given by labeling every $v \in V(H)$ with $\omega_H(v)$ and orienting $E(H)$ accordingly. Using the colors of the optimal coloring of H given by ω_H , we can find $\chi(H)$ mutually disjoint antichains of Q that cover Q , so by Corollary 3.11 we obtain that

$$(4.1) \quad e(Q) \geq \frac{1}{\chi(H)} \sum_{v \in V(H)} e(Q \setminus v).$$

Now, we note that each $Q \setminus v$ with $v \in V(H)$ is also induced by the respective restriction of ω to $V(H) \setminus v$, and that the comparability of $Q \setminus v$ is $H \setminus v$, and then each of the terms on the right hand side can be expanded similarly. Starting from $H = G$ above and noting the fact that $\varepsilon(G) = e(Q)$ for this case, we can expand the terms of 4.1 exhaustively to obtain the desired expression. \square

Corollary 4.4. *Let $G = G(V, E)$ be any graph on n vertices with chromatic number $k := \chi(G)$. Then $\varepsilon(G) \geq \frac{n!}{k^{n-k}k!}$.*

Proof. We can follow the proof of Theorem 4.3. This time, starting from $H = G$, Q will be a poset on V given by a minimal coloring of G , i.e. we color G using a minimal number of totally ordered colors and orient E accordingly. Then, $\varepsilon(G) \geq e(Q)$ and we can expand the right hand side of 4.1, but noting that $Q \setminus v$ can only be guaranteed to be partitioned into at most $\chi(G)$ antichains, and that the chromatic number of a graph is at most the number of vertices of that graph. \square

Noting that the number of cutsets is at least 2 in most cases, a similar argument to that of Theorem 4.3 implies:

Observation 4.5. *Let $G = G(V, E)$ be a connected graph. Then:*

$$\varepsilon(G) \leq \frac{1}{2} \sum_{v \in V} \varepsilon(G \setminus v).$$

Example 4.6. If $G = G(V, E)$ is the odd cycle on $2n + 1$ vertices, then for each $v \in V$ we have $\varepsilon(G \setminus v) = E_{2n}$, the $(2n)$ -th Euler number, and $\chi(G) = 3$, so $a_n := \frac{(2n+1)E_{2n}}{2} \geq \varepsilon(G) \geq b_n := \frac{(2n+1)!}{3^{2n-2} \cdot 3!}$. As n goes to infinity, then $\frac{a_n}{b_n} \sim \frac{4}{3\pi} \left(\frac{6}{\pi}\right)^{2n}$.

Other upper bounds can be obtained from rather different considerations.

Proposition 4.7. *Let $G = G(V, E)$ be a simple graph on n vertices. Then, $\varepsilon(G)$ is at most equal to the number of acyclic orientations of the edges of \bar{G} , the complement of G . Equality is attained if and only if G is a complete p -partite graph, $p \in [n]$.*

Proof. Let \bar{E} be the set of edges of \bar{G} , so that $E \sqcup \bar{E} = \binom{V}{2}$.

The inequality holds since two different linear extensions (understood as labelings of V with the totally ordered set $[n]$) of the same acyclic orientation of E induce different acyclic orientations of $\binom{V}{2} = E \sqcup \bar{E}$: As both induce the same orientation of E , they must induce different orientations of \bar{E} .

To prove the equality statement, first note that if G is not a complete p -partite graph, then there exist edges $\{a, b\}, \{a, c\} \in \bar{E}$ such that $\{b, c\} \in E$. Suppose that (b, c) is a directed edge in an optimal orientation O of E . Then, if we label the vertices of \bar{G} with the (totally ordered) set $[n]$ in such a way that $c < a < b$ comparing vertices according to their labels, our labeling induces an acyclic orientation of \bar{E} which cannot be obtained from a linear extension of O . Hence, $\varepsilon(G)$ is strictly less than the number of acyclic orientations of \bar{E} .

If G is a complete p -partite graph, then suppose that there exists an acyclic orientation \bar{O} of \bar{E} that cannot be obtained from a linear extension of O , where O is any optimal orientation of E . Then, in the union of the (directed) edges in both O and \bar{O} , we can find a directed cycle that uses at least one (directed) edge from both O and \bar{O} . Take one such directed cycle with minimal number of (directed) edges. As G is a comparability graph, then O is transitive, and so the directed cycle has the form $E_1 P_1 E_2 P_2 \dots E_m P_m$, where E_i is a directed edge in O , P_i is a directed path in \bar{O} , and $m \geq 1$. Let $E_1 = (a, b)$, and let (b, c) be the first directed edge in P_1 along the directed cycle. Since G is complete p -partite, then $\{a, c\} \in E$ because $\{b, c\} \in \bar{E}$. Since O is transitive, (a, c) must be a directed edge in O . However, this contradicts the minimality of the directed cycle. \square

4.3. Random graphs.

Changing the scope towards probabilistic models of graphs, specifically to $G_{n,p}$, we will obtain a tight concentration result for the random variable $\varepsilon(G)$ with $G \sim G_{n,p}$. The central idea of the argument will be to choose an acyclic orientation of a graph $G \sim G_{n,p}$ from a minimal proper coloring of its vertices. We expect this orientation to be nearly optimal.

Let us first recall two remarkable results that will be essential in our proof. The first one is a well-known result of Bollobás, later improved on by McDiarmid:

Theorem 4.8 (Bollobás (1988), McDiarmid (1990)). *Let $G \sim G_{n,p}$ with $0 < p < 1$, and define $b = \frac{1}{1-p}$. Then:*

$$\chi(G) = \frac{n}{2 \log_b n - 2 \log_b \log_b n + O(1)} \text{ a.s.,}$$

where $\chi(G)$ is the chromatic number of G .

To state the second result, we first need to introduce the concept of *entropy* of a *convex corner*, originally defined in Csiszár et al. (1990). We only present here the statement for the case of stable polytopes of graphs.

Definition 4.9. Let $G = G([n], E)$ be a simple graph, and let $\text{STAB}(B)$ be the stable polytope of G . Then, the entropy $H(G)$ of G is the quantity:

$$H(G) := \min_{a \in \text{STAB}(G)} - \sum_{i=1}^n \frac{1}{n} \log_2 a_i.$$

In 1995, Kahn and Kim proved certain bounds for the volumes of convex corners in terms of their entropies. One of them, when applied to stable polytopes, reads as follows:

Theorem 4.10 (Kahn and Kim (1995)). Let $G = G([n], E)$ be a simple graph, and let $\text{STAB}(G)$ be the stable polytope of G . Then:

$$n^n 2^{-nH(G)} \geq n! \text{Vol}(\text{STAB}(G)) \geq n! 2^{-nH(G)}.$$

Equipped now with these background results, the following is true:

Theorem 4.11. Let $G \sim G_{n,p}$ with $0 < p < 1$, $b = \frac{1}{1-p}$, and write $s = 2 \log_b n - 2 \log_b \log_b n$. Then:

$$\log_2 \varepsilon(G) \sim n \log_2 s \text{ holds a.s..}$$

Also, $\mathbf{E}[\log_2 \varepsilon(G)] \sim n \log_2 s$.

Proof. Let n tend to infinity. Consider the chromatic number of the graph $G \sim G_{n,p}$, and color G properly using $k = \chi(G)$ colors, say with color partition $a_1 + a_2 + \dots + a_k = n$. Then $\log_2 \varepsilon(G) \geq \log_2 a_1! + \dots + \log_2 a_k! \geq k \log_2 \lfloor \frac{n}{k} \rfloor!$. By Theorem 4.8, we know that $k = \frac{n}{s + O(1)}$ a.s., so:

$$(4.2) \quad \log_2 \varepsilon(G) \geq n \log_2 s - \frac{n}{\ln 2} + \frac{n}{2s} (\log_2 s) + O\left(\frac{n}{s}\right) \text{ a.s..}$$

We remark here that inequality 4.2 gives a slightly better bound than the one obtained directly from Corollary 4.4.

Now, the function $\log_2 \varepsilon$ satisfies the *edge Lipschitz condition* in the *edge exposure martingale* since addition of a single edge to G can alter ε by a factor of at most 2, so we can apply Azuma's inequality to obtain:

$$\Pr[|\log_2 \varepsilon(G) - \mathbf{E}[\log_2 \varepsilon(G)]| > n (\log_2 \log_b n)^{\frac{1}{2}}] < \frac{2}{\log_b n}.$$

Combining these two results, we see that:

$$\mathbf{E}[\log_2 \varepsilon(G)] \geq (n \log_2 s)(1 + o(1)),$$

and moreover, that $\log_2 \varepsilon(G) \sim \mathbf{E}[\log_2 \varepsilon(G)]$ a.s. holds.

The second necessary inequality comes, firstly, from using Observation 4.1, so that $\varepsilon(G) \leq n! \text{Vol}(\text{STAB}(G))$, and then from a direct application of Theorem 4.10. We obtain that $n(\log_2 n - H(G)) \geq \log_2 \varepsilon(G)$. Now, we further observe that for $a \in \text{STAB}(G)$, we have $\sum_i \frac{1}{n} a_i \leq \frac{1}{n} \alpha(G)$, and then:

$$H(G) = \sum_i \frac{1}{n} (-\log_2 a_i) \geq -\log_2 \left(\sum_i \frac{1}{n} a_i \right) \geq -\log_2 \frac{1}{n} \alpha(G) = \log_2 \frac{n}{\alpha(G)}.$$

A classic result of Grimmett and McDiarmid (1975) states that $\alpha(G) \leq s + c$ holds a.s., where $c = 2 \log_b \frac{e}{2} + 1$. Hence, a.s., $H(G) \geq \left(\log_2 \frac{n}{s+O(1)} \right) = \log_2 n - \log_2(s +$

$O(1))$, and then $n \log_2(s + O(1)) \geq \log_2 \varepsilon(G)$. From here, we directly obtain:

$$(4.3) \quad \log_2 \varepsilon(G) \leq n \log_2 s + O\left(\frac{n}{s}\right) \text{ a.s..}$$

Therefore, from inequalities 4.2 and 4.3:

$$\log_2 \varepsilon(G) = n \log_2 s + O(n) \text{ a.s..}$$

□

Calculating inequality 4.3 more precisely by dropping the O -notation and using Grimmett and McDiarmid's constant, we obtain:

Corollary 4.12. *Let $G \sim G_{n,p}$ with $0 < p < 1$, $b = \frac{1}{1-p}$ and $s = 2 \log_b n - 2 \log_b \log_b n$. Then, for large enough n :*

$$\frac{s^n}{n!} \cdot \left(\frac{1}{e}\right)^n \leq \text{Vol}(STAB(G)) \leq \frac{s^n}{n!} \cdot c^{n/s} \text{ a.s., where } c = 2 \left(\frac{e}{2}\right)^{2/(\log_2 b)}.$$

Corollary 4.13. *Let $G \sim G_{n,p}$ with $0 < p < 1$, $b = \frac{1}{1-p}$ and $s = 2 \log_b n - 2 \log_b \log_b n$. Then, for large enough n :*

$$\log_2 \left(\frac{n}{s}\right) + O\left(\frac{1}{s}\right) \leq H(G) \leq \log_2 \left(\frac{n}{s}\right) + \frac{1}{\ln 2} \text{ a.s..}$$

5. Further techniques.

In this section, we will see how the main problem has two more presentations as selecting a region in the graphical arrangement with maximal *fractional volume*, or as selecting a vertex of the graphical zonotope that is farthest from the origin in Euclidean distance.

Definition 5.1. *Consider a simple undirected graph $G = G([n], E)$. The graphical arrangement of G is the central hyperplane arrangement in \mathbb{R}^n given by:*

$$\mathcal{A}_G = \{x \in \mathbb{R}^n : x_i - x_j = 0, \forall \{i, j\} \in E\}.$$

The regions of the graphical arrangement \mathcal{A}_G with $G = G([n], E)$ are in one-to-one correspondence with the acyclic orientations of G . Moreover, the complete fan in \mathbb{R}^n given by \mathcal{A}_G is combinatorially dual to the *graphical zonotope* of G :

$$\mathcal{Z}_G^{\text{central}} := \sum_{\{i,j\} \in E} [e_i - e_j, e_j - e_i],$$

and there is a clear correspondence between the regions of \mathcal{A}_G and the vertices of $\mathcal{Z}_G^{\text{central}}$.

Following Klivans and Swartz (2011), we define the *fractional volume* of a region \mathcal{R} of \mathcal{A}_G to be: $\text{Vol}^\circ(\mathcal{R}) = \frac{\text{Vol}(B^n \cap \mathcal{R})}{\text{Vol}(B^n)}$, where B^n is the unit n -dimensional ball in \mathbb{R}^n .

With little work it is possible to say the following about these volumes:

Proposition 5.2. *Let $G = G([n], E)$ be an undirected simple graph, and let \mathcal{A}_G be its graphical arrangement. If \mathcal{R} is a region of \mathcal{A}_G and P is its corresponding partial order on $[n]$, then:*

$$\text{Vol}^\circ(\mathcal{R}) = \frac{e(P)}{n!}.$$

The problem of finding the regions of \mathcal{A}_G with maximal fractional volume is, intuitively, closely related to the problem of finding the vertices of $\mathcal{Z}_G^{\text{central}}$ that are farthest from the origin under some appropriate choice of metric. It turns out that, with Euclidean metric, a precise statement can be formulated when G is a comparability graph:

Theorem 5.3. *Let $G = G(V, E)$ be a comparability graph. Then, the vertices of the graphical zonotope of $\mathcal{Z}_G^{\text{central}}$ that have maximal Euclidean distance to the origin are precisely those that correspond to the transitive orientations of E , which in turn have maximal number $\varepsilon(G)$ of linear extensions.*

To prove Theorem 5.3, we first note that for a simple (undirected) graph $G = G(V, E)$, the vertex of $\mathcal{Z}_G^{\text{central}}$ corresponding to a given acyclic orientation of E is precisely the point:

$$(\text{outdeg}(v) - \text{indeg}(v))_{v \in V},$$

where $\text{outdeg}(\cdot)$ and $\text{indeg}(\cdot)$ are calculated using the given orientation.

We need to establish a preliminary lemma.

Lemma 5.4. *Let $G_o = G_o(V, E)$ be an oriented graph. Then,*

$$\frac{1}{2} \sum_{v \in V} (\text{indeg}(v) - \text{outdeg}(v))^2 = |E| + \text{tri}(G_o) + \text{incom}(G_o) - \text{com}(G_o),$$

where:

1. $\text{tri}(G_o)$ is the number of directed triangles $(u, v), (v, w), (u, w) \in E$.
2. $\text{incom}(G_o)$ is the number of triples $u, v, w \in V$ such that $(v, w), (w, v) \notin E$ but either $(u, v), (u, w) \in E$ or $(v, u), (w, u) \in E$.
3. $\text{com}(G_o)$ is the number of directed 2-paths $(u, v), (v, w) \in E$ such that $(u, w) \notin E$.

Proof. For $v \in V$, $\text{outdeg}(v)^2$ is equal to $\text{outdeg}(v)$ plus two times the number of pairs $u \neq w$ such that $(v, u), (v, w) \in E$, $\text{indeg}(v)^2$ is equal to $\text{indeg}(v)$ plus two times the number of pairs $u \neq w$ such that $(u, v), (w, v) \in E$, and $\text{outdeg}(v) \cdot \text{indeg}(v)$ is equal to the number of pairs $u \neq w$ such that $(u, v), (v, w) \in E$. If we add up these terms and cancel out terms in the case of directed triangles, we obtain the desired equality. \square

An important consequence of Lemma 5.4 is the following:

If $G = G(V, E)$ is a simple graph, all the acyclic orientations of E will not vary in their values of $\text{tri}(\cdot)$ and of $|E|$, which depend on G , but only in $\text{com}(\cdot)$ and $\text{incom}(\cdot)$. Moreover, $\text{com}(\cdot) + \text{incom}(\cdot)$ is equal to the number of 2-paths in G of the form $\{u, v\}, \{v, w\} \in E$ with $u \neq w$, so it is also independent of the choice of orientation for E .

Proof of Theorem 5.3. We apply Lemma 5.4 directly. Since G is a comparability graph, from Theorem 3.7, we know that the value of $\text{incom}(\cdot) - \text{com}(\cdot)$ will be maximized precisely on the transitive orientations of G , since all transitive orientations force $\text{com}(\cdot) = 0$. \square

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